QM3

Quantum Mechanics 3 notes by K. Sreeman Reddy.

We will use the metric signature $\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$

The Klein-Gordon equation

$(\Box + \left(\frac{mc}{\hbar}\right)^2)\psi(x) = 0,$ where $\Box = \partial^\mu \partial_\mu$

	Position space $x = (ct, \mathbf{x})$	Fourier transformation $\omega = E/\hbar, \quad \mathbf{k} = \mathbf{p}/\hbar$	Momentum space $p = (E/c, \mathbf{p})$
Separated time and space	$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)\psi(t, \mathbf{x}) = 0$	$\psi(t, \mathbf{x}) = \int \frac{\mathrm{d}\omega}{2\pi\hbar} \int \frac{\mathrm{d}^3 k}{(2\pi\hbar)^3} e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})}\psi(\omega, \mathbf{k})$	$E^2 = \mathbf{p}^2c^2 + m^2c^4$
Four-vector form	$(\Box + \left(\frac{mc}{\hbar}\right)^2)\psi(x) = 0,$	$\psi(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} e^{-ip\cdot x}\psi(p)$	$p^2 = +m^2$

Electromagnetic interaction

Let $D_\mu = \partial_\mu + \frac{iq}{c}A_\mu$ then under minimal coupling with A_μ

$$D_\mu D^\mu \phi = -(\partial_{ct} + \frac{iq}{\hbar}A_0)^2\phi + (\partial_i + \frac{iq}{\hbar}A_i)^2\phi = -\left(\frac{mc}{\hbar}\right)^2\phi$$
$$(p_\mu - \frac{q}{c}A_\mu)(p^\mu - \frac{q}{c}A^\mu) = m^2c^2\phi$$

Conserved current

$$\partial_\mu J^\mu(x) = 0, \quad J^\mu(x) \equiv \frac{i\hbar}{2m}\left(\phi^*(x)\partial^\mu\phi(x) - \phi(x)\partial^\mu\phi^*(x)\right) - \frac{q}{mc}A^\mu\phi^*\phi$$

Non-relativistic limit

$\psi(x,t) = \phi(x,t)\,e^{-\frac{i}{\hbar}mc^2 t} \quad \text{where} \quad \phi(x,t) = u_E(x)e^{-\frac{i}{\hbar}E' t}.$
Since $\left|i\hbar\frac{\partial\phi}{\partial t}\right| = E'\phi \ll mc^2\phi$ and $|qA_0\phi| << mc^2\phi$ in the non-relativistic limit

$$i\hbar\frac{\partial\phi}{\partial t} = \left[\frac{1}{2m}(\vec{p} - \frac{q}{c}\vec{A})^2 + mc^2 + qA^0\right]\phi$$

we get back the classical Pauli field.

The Dirac equation

$i\hbar\gamma^\mu\partial_\mu\psi - mc\psi = 0$
In Dirac representation

$$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$$

In Weyl (chiral) representation

$$\gamma^0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix}$$

In Weyl (chiral) representation (alternate form)

$$\gamma^0 = \begin{pmatrix} 0 & -I_2 \\ -I_2 & 0 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$$

In Majorana representation

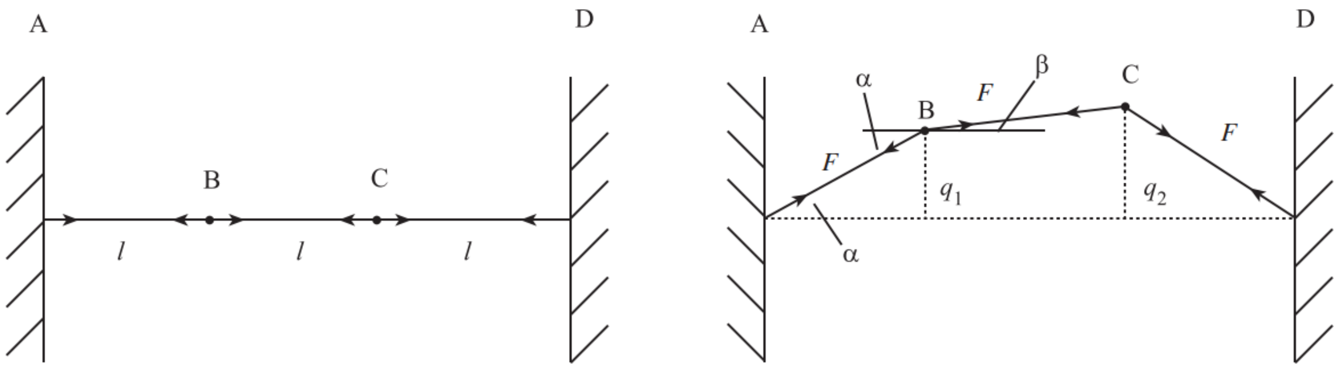
$$\begin{aligned} \gamma^0 &= \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, & \gamma^1 &= \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix}, & \gamma^2 &= \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \\ \gamma^3 &= \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix}, & \gamma^5 &= \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix}, & C &= \begin{pmatrix} 0 & -i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix}, \end{aligned}$$

Classical Field Theory

In a classical string, there are uncountable number of particles but only countable modes **if we fix some boundary conditions**. But if no boundary conditions we will have uncountable modes.

Introduction

A vibrating system



The below 2 equations are linear because we neglected the higher order terms. But quantum mechanics is believed to be **a linear theory without any approximation**.

$$m\ddot{q}_1 = -\partial V/\partial q_1$$
$$m\ddot{q}_2 = -\partial V/\partial q_2$$
$$V = k\left(q_1^2 + q_2^2 - q_1 q_2\right)$$

Now observe that if we define

$$Q_1 = (q_1 + q_2)/\sqrt{2} \quad Q_2 = (q_1 - q_2)/\sqrt{2}$$

then for $\omega_1 = \sqrt{\frac{k}{m}}$ and $\omega_2 = \sqrt{\frac{3k}{m}}$

$$m\ddot{Q}_1 = -\partial V/\partial Q_1$$
$$m\ddot{Q}_2 = -\partial V/\partial Q_2$$
$$V \equiv V(Q_1, Q_2) = \frac{1}{2}m\omega_1^2 Q_1^2 + \frac{1}{2}m\omega_2^2 Q_2^2$$

A remarkable thing has happened: the two combinations $q_1 + q_2$ and $q_1 - q_2$ of the original coordinates satisfy uncoupled equations. These are called **normal modes** or **modes**.

- In general the system is in ‘a superposition of modes’.
- Modes do not interact.
- The simple change of variables $(q_1, q_2) \to (Q_1, Q_2)$ does remove the $q_1 q_2$ coupling, this would not be the case if, say, cubic terms in V were to be considered.

Increase the degrees of freedom to N from 2

In general we can find the **mode coordinates** or **normal coordinates**

$$Q_r = \sum_{s=1}^{N} a_{rs} q_s$$

such that

$$E = \sum_{r=1}^{N} \frac{1}{2}m\dot{q}_r^2 + V(q_1, \ldots, q_r)$$
$$E = \sum_{r=1}^{N} \left[\frac{1}{2}m\dot{Q}_r^2 + \frac{1}{2}m\omega_r^2 Q_r^2\right]$$

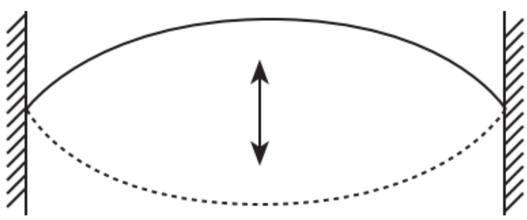
Quantizing it

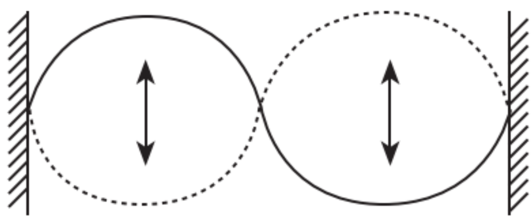
$$E = \sum_{r=1}^{N} \left(n_r + \frac{1}{2}\right)\hbar\omega_r$$

We forget about the original N degrees of freedom $q_1, q_2, \ldots, q_N$ and the original N ‘atoms’, which indeed are only remembered in the above equation via the fact that there are N different mode frequencies ωr. Instead we concentrate on the quanta and treat them as ‘things’ which really determine the behaviour of our quantum system.

For the state characterized by $(n_1, n_2, \ldots, n_N)$ there
are n_1 quanta of mode 1 (frequency ω_1), n_2 of mode 2, $\ldots$ and n_N of mode N.

Classical string





Let $N \to \infty$ and $Na = l$

$$\frac{1}{c^2}\frac{\partial^2 \phi(x,t)}{\partial t^2} = \frac{\partial^2 \phi(x,t)}{\partial x^2}$$

$$\phi(x,t)=\sum_{r=1}^{\infty} A_r(t)\sin\left(\frac{r\pi x}{\ell}\right)$$

$$E=\int_0^{\ell}\left[\frac{1}{2}\rho\left(\frac{\partial\phi}{\partial t}\right)^2+\frac{1}{2}\rho c^2\left(\frac{\partial\phi}{\partial x}\right)^2\right]\mathrm{d}x$$

$$E=(\ell/2)\sum_{r=1}^{\infty}\left[\frac{1}{2}\rho\dot{A}_r^2+\frac{1}{2}\rho\omega_r^2A_r^2\right]$$

Quantizing it

$$E=\sum_{r=1}^{\infty}\left(n_r+\frac{1}{2}\right)\hbar\omega_r$$

We remark that as $l\to\infty$, the mode sum will be replaced by an integral over a continuous frequency variable.

Lagrange formulation

$$\frac{\partial\mathcal{L}}{\partial\phi^a}-\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi^a\right)}\right)=0$$

$$S=\int\mathrm{d}t\int\mathrm{d}^3x\mathcal{L}=\int\mathrm{d}^4x\mathcal{L}$$

$$\mathcal{L}'=\mathcal{L}+\partial_\mu K^\mu(\Phi)$$

The above Lagrangian also is equivalent to $\mathcal{L}$.

Hamilton formulation

For many fields $\phi_i(\mathbf{x},t)$ and their conjugates $\pi_i(\mathbf{x},t)$ the Hamiltonian density is a function of them all:

$$\mathcal{H}(\phi_1,\phi_2,\ldots,\pi_1,\pi_2,\ldots,\mathbf{x},t)=\sum_i\dot{\phi}_i\pi_i-\mathcal{L}(\phi_1,\phi_2,\ldots\nabla\phi_1,\nabla\phi_2,\ldots,\partial\phi_1/\partial t,\partial\phi_2/\partial t,\ldots,\mathbf{x},t)\,.$$

where each conjugate field is **defined** with respect to its field,

$$\pi_i(\mathbf{x},t)=\frac{\partial\mathcal{L}}{\partial\dot{\phi}_i}$$

$$H=\int\mathcal{H}\;d^3x$$

Hamiltonian field equations:

$$\dot{\phi}_i=+\frac{\delta\mathcal{H}}{\delta\pi_i}\,,\quad\dot{\pi}_i=-\frac{\delta\mathcal{H}}{\delta\phi_i}$$

$$\frac{\delta}{\delta\phi_i}=\frac{\partial}{\partial\phi_i}-\partial_\mu\frac{\partial}{\partial(\partial_\mu\phi_i)}$$

Noether's theorem

If $\Delta\mathcal{L}$, $\Delta\phi$ and Δx are defined by $\mathcal{L}(x)\to\mathcal{L}(x)+\epsilon\Delta\mathcal{L}$ (we know that $\Delta\mathcal{L}=\partial_\mu K^\mu$ for some K^μ), $\phi\to\phi+\epsilon\Delta\phi$ and $x\to x+\epsilon\Delta x$ then the conserved current current defined by

$$j^\mu=\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\Delta\phi-K^\mu$$

satisfies $\partial_\mu j^\mu=0$. (j^μ is unique up to a multiplicative constant)
If the symmetry involves more than one field, the conserved current is

$$j^\mu=\sum_i\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi_i\right)}\Delta\phi_i-K^\mu$$

The above conservation law implies that the **Noether charge** over all space is conserved

$$Q = \int d^3\mathbf{x} j^0$$

is constant in time: $\frac{dQ}{dt} = 0$.
OR:

$$T^{\mu\nu} = \frac{\partial \mathscr{L}}{\partial\left(\partial_\mu \Phi^A\right)} \partial^\nu \Phi^A - \eta^{\mu\nu} \mathscr{L}$$

which is called the stress-energy tensor. We can now define a current

$$J^\mu = \frac{\partial \mathscr{L}}{\partial\left(\partial_\mu \Phi^A\right)} \delta \Phi^A - T^{\mu\nu} \delta x_\nu$$

and write

$$\delta \mathscr{A} = \int_\Omega d^4 x \partial_\mu J^\mu$$

This relation holds for arbitrary variations of the fields and coordinates provided the equations of motion are satisfied.

Conservation laws

The 1st 4 symmetries together form the Poincaré group

Conservation Law	Respective Noether symmetry invariance	Number of dimensions
Conservation of mass-energy	Time-translation invariance	1: translation along time axis
Conservation of linear momentum	Space-translation invariance	3: translation along x,y,z directions
Conservation of angular momentum	Rotation invariance	3: rotation about x,y,z axes
Conservation of CM (center-of-momentum) velocity	Lorentz-boost invariance	3: Lorentz-boost along x,y,z directions
Conservation of electric charge	$U(1)$ Gauge invariance	$1 \otimes 4$: scalar field (1D) in 4D spacetime (x,y,z + time evolution)
Conservation of color charge	$SU(3)$ Gauge invariance	3: r,g,b
Conservation of weak isospin	$SU(2)_L$ Gauge invariance	1: weak charge
Conservation of probability	Probability invariance	$1 \otimes 4$: total probability always = 1 in whole x,y,z space, during time evolution

The free scalar quantum field

$$\mathcal{L}_{\mathrm{KG}} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$$

Canonical or 2nd Quantization

$$[\hat{\phi}(\vec{x}, t), \hat{\pi}(\vec{y}, t)] = \mathrm{i} \delta^3(\vec{x} - \vec{y})$$

Heisenberg picture

$$\dot{\hat{\phi}}(x) = \mathrm{i}[\hat{H}, \hat{\phi}(x)]$$
$$\dot{\hat{\pi}}(x) = \mathrm{i}[\hat{H}, \hat{\pi}(x)]$$

Fourier decomposition of the field

The four-dimensional analogue of the Fourier expansion of the field ϕ takes the form

$$\hat{\phi}(x) = \int_{-\infty}^{\infty} \frac{\mathrm{d}^3 k}{(2\pi)^3 \sqrt{2\omega}} \left[\hat{a}(k) \mathrm{e}^{-\mathrm{i}k \cdot x} + \hat{a}^\dagger(k) \mathrm{e}^{\mathrm{i}k \cdot x}\right]$$

with a similar expansion for the **conjugate momentum** $\hat{\pi} = \dot{\hat{\phi}}$:

$$\hat{\pi}(x) = \int_{-\infty}^{\infty} \frac{\mathrm{d}^3 k}{(2\pi)^3 \sqrt{2\omega}} (-\mathrm{i}\omega) \left[\hat{a}(k)\mathrm{e}^{-\mathrm{i}k\cdot x} - \hat{a}^\dagger(k)\mathrm{e}^{\mathrm{i}k\cdot x}\right]$$

Here $k \cdot x$ is the four-dimensional dot product $k \cdot x = \omega t - \boldsymbol{k} \cdot \boldsymbol{x}$, and $\omega = +\left(\boldsymbol{k}^2 + m^2\right)^{1/2}$. A **positive-frequency** solution of the field equation has as its coefficient the operator that **destroys** a particle in that single-particle wavefunction. A **negative-frequency** solution of the field equation, being the Hermitian conjugate of a positive-frequency solution, has as its coefficient the operator that **creates** a particle in that positive-energy single-particle wavefunction.

$$\hat{n}(k) = \hat{a}^\dagger(k)\hat{a}(k)$$

The Hamiltonian is found to be

$$\hat{H}_{\mathrm{KG}} = \int \mathrm{d}^3 x \hat{\mathcal{H}}_{\mathrm{KG}} = \int_{-\infty}^{\infty} \mathrm{d}^3 x \frac{1}{2}\left[\hat{\pi}^2 + \nabla\hat{\phi}\cdot\nabla\hat{\phi} + m^2\hat{\phi}^2\right]$$

and this can be expressed in terms of the $\hat{a}$'s and the $\hat{a}^\dagger$'s using the expansion for $\hat{\phi}$ and $\hat{\pi}$ and the commutator

$$\left[\hat{a}(k), \hat{a}^\dagger\left(k'\right)\right] = (2\pi)^3 \delta^3\left(\boldsymbol{k} - \boldsymbol{k}'\right)$$

with all others vanishing. The result is, as expected,

$$\hat{H}_{\mathrm{KG}} = \frac{1}{2}\int \frac{\mathrm{d}^3 \boldsymbol{k}}{(2\pi)^3}\left[\hat{a}^\dagger(k)\hat{a}(k) + \hat{a}(k)\hat{a}^\dagger(k)\right]\omega$$

and, **normally ordering** (operators rearranged with all creation operators on the left) as usual, we arrive at

$$\hat{H}_{\mathrm{KG}} = \int \frac{\mathrm{d}^3 \boldsymbol{k}}{(2\pi)^3} \hat{a}^\dagger(k)\hat{a}(k)\omega$$

$$\vec{P} = -\int d^3 x \pi \vec{\nabla}\phi = \int \frac{d^3 p}{(2\pi)^3} \vec{p} a_{\vec{p}}^\dagger a_{\vec{p}}$$

Creation and annihilation operators

$$a_{\mathbf{p}} = \frac{i}{\sqrt{2\omega_{\mathbf{p}}}} \int d^3\mathbf{x}\left[\Pi(\mathbf{x}) - i\omega_{\mathbf{p}}\phi(\mathbf{x})\right] e^{-i\mathbf{p}\cdot\mathbf{x}}$$

$$a_{\mathbf{k}}^\dagger = \frac{-i}{\sqrt{2\omega_{\mathbf{k}}}} \int d^3\mathbf{y}\left[\Pi(\mathbf{y}) + i\omega_{\mathbf{k}}\phi(\mathbf{y})\right] e^{i\mathbf{k}\cdot\mathbf{y}}$$

$$\left[\hat{H}, a_{\mathbf{p}}^\dagger\right] = \omega_{\mathbf{p}} a_{\mathbf{p}}^\dagger$$

$$\left[\hat{H}, a_{\mathbf{p}}\right] = -\omega_{\mathbf{p}} a_{\mathbf{p}}$$

$$\overbrace{[H,[H,[\cdots[H}^{n\text{ times}}, a_{\mathbf{p}}]\cdots]]] = (-\omega_{\mathbf{p}})^n a$$

$$\overbrace{[\hat{H},[\hat{H},[\cdots[\hat{H}}^{n\text{ times}}, a_{\mathbf{p}}^\dagger]\cdots]]] = (\omega_{\mathbf{p}})^n a$$

$$\hat{H} a_{\mathbf{p}} = a_{\mathbf{p}}\left(\hat{H} - E_{\mathbf{p}}\right)$$
$$\hat{H}^n a_{\mathbf{p}} = a_{\mathbf{p}}\left(\hat{H} - E_{\mathbf{p}}\right)^n$$

$$e^{i\hat{H}t} a_{\mathbf{p}} e^{-i\hat{H}t} = a_{\mathbf{p}} e^{-iE_{\mathbf{p}}t},$$
$$e^{i\hat{H}t} a_{\mathbf{p}}^\dagger e^{-i\hat{H}t} = a_{\mathbf{p}}^\dagger e^{iE_{\mathbf{p}}t}$$

$$e^{-i\hat{\mathbf{P}}\cdot\mathbf{x}} a_{\mathbf{p}} e^{i\hat{\mathbf{P}}\cdot\mathbf{x}} = a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}}$$

$$e^{-i\hat{\mathbf{P}}\cdot\mathbf{x}} a_{\mathbf{p}}^\dagger e^{i\hat{\mathbf{P}}\cdot\mathbf{x}} = a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}$$

$$\begin{aligned}\phi(x) &= e^{i(\hat{H}t - \hat{\mathbf{P}}\cdot\mathbf{x})}\phi(0)e^{-i(\hat{H}t - \hat{\mathbf{P}}\cdot\mathbf{x})} \\ &= e^{iP\cdot x}\phi(0)e^{-iP\cdot x}\end{aligned}$$

Wavefunctions

$$\left\langle 0|\hat{\phi}(x,t)|k'\right\rangle = \left\langle 0\left|\int \frac{\mathrm{d}k}{2\pi\sqrt{2\omega}}\left[\hat{a}(k)\mathrm{e}^{\mathrm{i}kx-\mathrm{i}\omega t}+\hat{a}^{\dagger}(k)\mathrm{e}^{-\mathrm{i}kx+\mathrm{i}\omega t}\right]N\hat{a}^{\dagger}\left(k'\right)\right|0\right\rangle$$
$$\left\langle 0\left|\int \frac{N\,\mathrm{d}k}{2\pi\sqrt{2\omega}}\left[\hat{a}^{\dagger}\left(k'\right)\hat{a}(k)+2\pi\delta\left(k-k'\right)\right]\mathrm{e}^{\mathrm{i}kx-\mathrm{i}\omega t}\right|0\right\rangle = N\frac{\mathrm{e}^{\mathrm{i}k'x-\mathrm{i}\omega' t}}{\sqrt{2\omega'}}$$
$$\left\langle 0|\hat{\phi}(x,t)|0\right\rangle = 0$$

Thus we discover that the vacuum to one-particle matrix elements of the field operators are just the familiar wavefunctions of single-particle quantum mechanics.

$$\langle\mathbf{q}|\phi(\mathbf{x})|0\rangle = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-i\mathbf{p}\cdot\mathbf{x}}\langle\mathbf{q}|\mathbf{p}\rangle = e^{-i\vec{q}\cdot\vec{x}}$$

In the language of second quantization, eiq·x tells us how much amplitude there is in
the qth momentum mode if we create a scalar particle at spacetime point x.

Normalisation

- $\mathrm{d}^4p\delta(p^2-m^2)\theta(p^0)$ is Lorentz invariant.
- Using $\delta[f(x)] = \sum_{\{x|f(x)=0\}}\frac{1}{|f'(x)|}\delta(x)$ we get

$$\delta\left(p^2-m^2\right)\theta\left(p_0\right) = \frac{1}{2E_p}\delta\left(p_0-E_p\right)\theta\left(p_0\right)$$

$$|\mathbf{p}\rangle = \sqrt{2E_{\mathbf{p}}}a_{\mathbf{p}}^{\dagger}|0\rangle$$

$$\langle\mathbf{p}\mid\mathbf{k}\rangle = 2E_{\mathbf{p}}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{k})$$

$$\mathbf{1}_{1-\text{ particle }} = \int \frac{d^3\mathbf{p}}{(2\pi)^3}|\mathbf{p}\rangle\frac{1}{2E_{\mathbf{p}}}\langle\mathbf{p}|$$

Causality

If $\phi(x),\phi(y)$ vanishes, one measurement cannot affect the other. In fact, if the commutator vanishes for $(x-y)^2<0$, causality is preserved quite generally, since
commutators involving any function of $\phi(x)$, including $\pi(x)=\dfrac{\partial\phi}{\partial t}$, would also have to vanish.

$$\begin{aligned}[\phi(x),\phi(y)] &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\int\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{q}}}} \\ &\times\left[\left(a_{\mathbf{p}}e^{-ip\cdot x}+a_{\mathbf{p}}^{\dagger}e^{ip\cdot x}\right),\left(a_{\mathbf{q}}e^{-iq\cdot y}+a_{\mathbf{q}}^{\dagger}e^{iq\cdot y}\right)\right] \\ &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\left(e^{-ip\cdot(x-y)}-e^{ip\cdot(x-y)}\right) \\ &= D(x-y)-D(y-x)\end{aligned}$$

- $[\phi(x),\phi(y)]$ is Lorentz invariant and it is a complex number.
- When $(x-y)^2<0$ we can always change our frame such that $(x-y)\rightarrow-(x-y)$ and the commutator will be 0. This we can't do if $(x-y)^2>0$.

In quantum field theory, causality requires that every particle have a corresponding antiparticle with the same mass and opposite quantum numbers (in this case electric charge). For the real-valued Klein-Gordon field, the particle is its own antiparticle.

Interacting Scalar Fields