Introduction to Numerical Analysis notes

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x_i is the interpolation point.

$\mathbb{P}_n$ = set of all polynomials with degree $\leq n$

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→ Joseph-Louis Lagrange's Interpolation theorem

$n+1$ distinct points → Unique polynomial interpolant in $\mathbb{P}_n$

Proof: 1) Uniqueness, $w(x) = p(x) - q(x) \in \mathbb{P}_n$

$\Downarrow$

Contradiction $\Leftarrow$ $\boxed{n+1}$ roots $\Leftarrow$ x_i, $i = 0, 1, \ldots n$ are roots

2) Existence

$$L_k^n(x) = \frac{(x-x_0)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$$

Lagrange polynomials $\Leftarrow$ $\boxed{p(x) = \sum_{k=0}^{n} f_k L_k^n(x)}$ $\boxed{k = 0, 1, \ldots n}$

$$L_k^n(x) = \prod_{j=0, j\neq k}^{n} \left(\frac{x - x_j}{x_k - x_j}\right)$$

→ Similarily Lagrange's Zero One Blocks → for sequences

Matrix method

$$AX = B$$

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & & & \\ & & & & \\ 1 & x_n & \cdots & & x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{bmatrix}$$

→ Distance: If f, g are defined on $[a,b]$

$\max\limits_{x \in [a,b]} |f(x) - g(x)|$ is the distance.

→ $C[a,b]$ – set of $\boxed{\text{all}}$ continous functions on $[a,b]$
$C^k[a,b]$ → " k times continously differentiable

$$\|f\| := \max_{x \in [a,b]} |f(x)|$$

→ Called Norm on Maximum Norm

Weisterass Approximation theorem

$f \in C[a,b] \Rightarrow \forall \epsilon > 0, \exists$ a polynomial p such that $\boxed{\|f - p\| < \epsilon}$ (p depends on f and ϵ)

Error equation theorem

$f \in C^{n+1}[a,b]$

Let $p \in P_n$ s.T $p(x_i) = f(x_i)$ $i = 0, 1, \cdots n$

then $\forall x \in [a,b]$, $\exists$ $\xi \in [a,b]$ s.t.

$$\boxed{f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{k=0}^{n} (x - x_k)}$$

depends on x.

<u>Proof</u> Fix $x \in [a,b]$

$$\varphi(s) := (f(s) - p(s)) \prod_{k=0}^{n} (x - x_k) - (f(x) - p(x)) \prod_{k=0}^{n} (s - x_k)$$

Rolle's theorem $\Rightarrow$ $\varphi^{(n+1)}$ has atleast 1 root ξ.

$$\Rightarrow f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{k=0}^{n} (x - x_k)$$

$$\max_{x \in [a,b]} |f(x) - p(x)| \leq \frac{1}{(n+1)!} \|f^{(n+1)}\| \max_{x \in [a,b]} \prod_{k=0}^{n} |x - x_k|$$

For Runge's example ($f(x) = \frac{1}{1+x^2}$ $x \in [-5, 5]$) use Chebyshev interpolation points

$$x_j = 5 \cos\left(\frac{(n-j)\pi}{n}\right) \quad j = 0, 1, \ldots n$$

These points imply $\Rightarrow \left\| \prod_{k=0}^{n} x - x_k \right\|_{\min} = \left(\frac{b-a}{2}\right)^{n+1} \times \frac{1}{2^n}$

<u>Newton's divided differences</u>

$$p(x) = f[x_0] + f[x_0, x_1](x - x_0) + \cdots + f[x_0, x_1, \ldots x_n] \prod_{j=0}^{n-1} (x - x_j)$$

$$f[x_\vartheta] := f(x_\vartheta)$$

$$f[x_\vartheta, \ldots x_{\vartheta+j}] := \frac{f[x_{\vartheta+1}, \ldots x_{\vartheta+j}] - f[x_\vartheta, \ldots x_{\vartheta+j-1}]}{x_{\vartheta+j} - x_\vartheta}$$

Ordering is not important

$$(fg)[x_0, \dots x_n] = \sum_{q=0}^{n} f[x_0, \dots x_q]\, g[x_q, \dots x_n]$$

$$f[x_0, x_1, \dots x_n] = \sum_{j=0}^{n} \frac{f(x_j)}{\prod_{k \in \{0,\dots n\} \setminus \{j\}} (x_j - x_k)}$$

Divided difference:

$f[x_0, x_1, \dots x_n]$ = Coefficient of x^n in $p \in P_n$

where p is the interpolant in P_n

$$\begin{array}{ccccc}
f[x_0] & & & & \\
 & \searrow & f[x_0, x_1] & & \\
f[x_1] & \nearrow\searrow & & \searrow & f[x_0, x_1, x_2] \\
 & & f[x_1, x_2] & \nearrow & \\
f[x_2] & \nearrow & & &
\end{array}$$

$$\boxed{f[x_0, x_1, x_2, \dots x_n] = \frac{f^{(n)}(\xi)}{\lfloor n}, \quad \xi \in (x_0, x_n)}$$

~~Lagrange's recipe~~

	+	−	×	÷
Lagrange's	n	$2n(n+1)$	$2n^2+n-1$	$n+1$
D.D	n	$\frac{3n(n+1)}{2}$	$\frac{n(n+1)}{2}$	$\frac{n(n+1)}{2}$

Note: If $q^{(n)}$ is a sequence of polynomials

s.t $\lim_{n\to\infty} \|f - q^{(n)}\| = 0 \Rightarrow \lim_{n\to\infty}$ Degree of $q^{(n)} = \infty$

f: not polynomial

$\Rightarrow$ But if $p^{(n)}$ is a sequence of interpolation ~~points~~ polynomials $\lim_{n\to\infty} \|f - p^{(n)}\| \neq 0$. (They may converge for some particular type of points)

Splines

Polynomial in P_m

$x_0 \quad x_1 \quad x_n$

$m \le n$

$$\Psi(x) = \begin{cases} a_0^{(1)} + a_1^{(1)} x + a_2^{(1)} x^2 + \cdots a_m^{(1)} x^m & \text{on } [x_0, x_1] \\ \vdots \\ a_0^{(n)} + a_1^{(n)} x + \cdots - a_m^{(n)} x^m & \text{on } [x_{n-1}, x_n] \end{cases}$$

Continuity $\quad 2n$ conditions

1st diff $\quad n-1$

$\vdots$

$\boxed{m-1}$th diff $\quad n-1$

$$2n + (m-1)(n-1)$$

$$= 2n + mn - n - m + 1$$

$$= \boxed{mn + n - m + 1}$$

We need $(m+1)\,n$

$\Rightarrow$ We need $(m-1)$ more conditions

$m=1 \Rightarrow$ no conditions

$m=3 \Rightarrow$ 2 conditions

$$\boxed{h_i = x_i - x_{i-1}}$$

$$h = \max_{1 \le i \le n} h_i$$

Error linear splines

$f \in C^2[a,b]$

$$h = \max_{1 \le i \le n} (x_i - x_{i-1})$$

$$\boxed{\| f - S_L \| \le \frac{h^2}{8} \| f'' \|}$$

Cubic splines

2 extra conditions

$$\boxed{S_0''(x_0) = S_{n-1}''(x_n) = 0}$$

$$S_i = a_i(x-x_i)^3 + b_i(x-x_i)^2 + c_i(x-x_i) + d_i$$

$$x \in [x_i, x_{i+1}]$$

Natural cubic splines with $h_i = h_j \ \forall\, i,j$

$$\sigma_i = S''(x_i)$$

$$d_i = f_i$$

$$b_i = \frac{\sigma_i}{2}$$

$$\boxed{b_0 = b_n = 0}$$

$$a_i = \frac{\sigma_{i+1} - \sigma_i}{6h}$$

$$\boxed{c_i = \frac{f_{i+1} - f_i}{h} - \frac{h}{6}(2\sigma_i + \sigma_{i+1})}$$

$$\sigma_{i-1} + 4\sigma_i + \sigma_{i+1} = \frac{6}{h^2}(f_{i-1} - 2f_i + f_{i+1})$$

$$\begin{pmatrix} 4 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 4 & 1 & & & & \\ 0 & 1 & 4 & & & & \\ \vdots & & & & & & \\ 0 & & & & 4 & 1 & 0 \\ 0 & 0 & & & 1 & 4 & 1 \\ 0 & 0 & 0 & & 0 & 1 & 4 \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \vdots \\ \sigma_{n-2} \\ \sigma_{n-1} \end{pmatrix} = \frac{6}{h^2} \begin{pmatrix} f_0 - 2f_1 + f_2 \\ f_1 - 2f_2 + f_3 \\ \vdots \\ f_{n-2} - 2f_{n-1} + f_n \end{pmatrix}$$

↓ ~~Diagnol~~ Diagonally dominant $\Rightarrow$

Invertible

Natural cubic spline error

$$\cancel{\|f-s\| \leq \frac{h^4}{8}\|f^{(iv)}\|}$$

$$\boxed{\|f-s\| \leq \|f''-s''\|\frac{h^2}{8}}$$

Proof $g := f - s_i$ on $[x_i, x_{i+1}]$

$\Rightarrow g(x_i) = 0,\ g(x_{i+1}) = 0 \Rightarrow$

Linear spline for g is just 0 function.

Using linear error $\|g-0\| \leq \frac{h^2}{8}\|g''\|$

$\Rightarrow \|f-s_i\| \leq \|f''-s_i''\|\frac{h^2}{8}$

$\Rightarrow \boxed{\|f-s\| \leq \frac{h^2}{8}\|f''-s''\|}$

2A Numerical Integration

$$\int_a^b f(x)\,dx \approx \int_a^b p(x)\,dx$$

$\downarrow$ Interpolotion fon

$a = x_0 < x_1 < x_2 < \cdots < x_n = b$

~~$x_i = a + ih,\ i = 0,1$~~

Newton ~~and Cotes integration~~ formulas

take

Newton and Cotes formula for integration

Take equally spaced points $x_i := a + ih$ $\boxed{0 \le i \le n}$

$$n = \frac{b-a}{h} \quad \text{or} \quad \boxed{nh = b-a}$$

$$p(x) = \sum_{i=0}^{n} f(x_i) L_i(x), \quad L_i(x) = \prod_{k=0, k\neq i}^{n} \left(\frac{x - x_k}{x_i - x_k}\right)$$

$$t = \frac{x-a}{h} = n\left(\frac{x-a}{b-a}\right) \in [0, n]$$

$$L_i(x) = \boxed{\prod_{k=0, k\neq i}^{n} \left(\frac{t-k}{i-k}\right) := \ell_i(t)}$$

$$\int_a^b f(x)\,dx \approx \int_a^b \sum_{i=0}^{n} f(x_i) L_i(x)\,dx$$

$$= \sum_{i=0}^{n} f(x_i) \int_a^b L_i(x)\,dx$$

$$\boxed{\int_a^b f(x) \simeq h \sum_{i=0}^{n} w_i f(x_i)} \quad \boxed{w_i := \int_0^n \ell_i(t)\,dt}$$

depends only on n.

$$\boxed{\sum_{i=0}^{n} w_i = n} \quad \boxed{w_k = w_{n-k}}$$

$n=1$	Trapezoidal rule	$\frac{h}{2}(f_0 + f_1)$
$n=2$	Simpson's rule	$\frac{h}{3}(f_0 + 4f_1 + f_2)$
$n=3$	Simpson's $\frac{3}{8}$ rule	$\frac{3h}{8}(f_0 + 3f_1 + 3f_2 + f_3)$
$n=4$	~~Boole's~~ Milne's rule	$\frac{2h}{45}(7f_0 + 32f_1 + 12f_2 + 32f_3 + 7f_4)$

2B Error in Newton-Cotes formula

$$\mathcal{I}_f = \int_a^b f(x)dx \qquad \mathcal{I}_{p_n} = \int_a^b p_n(x)dx$$

If $f \in C^{n+1}[a,b]$, $f(x) - p_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \prod_{i=0}^{n} (x - x_i)$

$\xi \in [a,b]$

$$|\mathcal{I}_f - \mathcal{I}_{p_n}| = \left| \int_a^b (f(x) - p_n(x))dx \right| \le \int_a^b |f(x) - p_n(x)|dx$$

$$\boxed{|\mathcal{I}_f - \mathcal{I}_{p_n}| \le \frac{\|f^{(n+1)}\|}{(n+1)!} \int_a^b \prod_{i=0}^{n} |x - x_i| dx}$$

For $n=1$

$\Rightarrow |\mathcal{I}_f - \mathcal{I}_{p_1}| \le \frac{\|f''\|}{2} \frac{(b-a)^3}{6}$

For $n=2$

$|\mathcal{I}_f - \mathcal{I}_{p_2}| \le \frac{\|f''''\|}{196} (b-a)^4$

2C In Runge example, Newton Cotes method doesnot converge as $n \to \infty$

Gaussian quadrature

$$G_n(f) = \sum_{i=0}^{n} w_i f(x_i)$$

Not equally spaced

$$W_i = \int_a^b (L_i(x))^2 dx = \int_a^b \prod_{k=0, k\neq i}^{n} \left(\frac{x - x_k}{x_i - x_k}\right)^2 dx$$

Weights non negative

Let $f \in C[a,b] \Rightarrow \boxed{\lim_{n\to\infty} |G_n(f) - \mathcal{I}_f| = 0}$

Composite Newton Cotes (converges)

Composite trapezoidal

$$C_{P_1}(f) = h\left(\tfrac{1}{2}f(a) + f(a+h) + \cdots + f(a+(m-1)h) + \tfrac{1}{2}f(b)\right)$$

$$h = \frac{b-a}{m}$$

$\boxed{h = \frac{b-a}{nm}}$

Error: $\boxed{\left|C_{P_1}(f) - I_f\right| \le m \times \frac{1}{12}\|f''\| h^3 = \frac{b-a}{12}\|f''\| h^2}$

Composit ~~sy~~ simpson's

$$C_{P_2}(f) = \frac{h}{3}\left(f(a) + 4f(a+h) + 2f(a+2h) + \cdots + 2f(a+(2m-2)h) + 4f(a+(2m-1)h) + f(b)\right)$$

$\boxed{h = \frac{b-a}{2m}}$ $\left|C_{P_2}(f) - I_f\right| \le \frac{(b-a)}{180}\|f''''\| h^3$

3A ODEs

$y' = \sqrt{y} \qquad y(0) = 0$

Solution: $y(t) = \begin{cases} \frac{(t-k)^2}{4} & t \ge k \\ 0 & t < k \end{cases}$ $\boxed{\infty}$

or $\boxed{y(t) = 0 \quad \forall y}$

$y' = \sqrt{y}$, $y(0) \neq 0 \Rightarrow$ Unique solution.

Notation: $\boxed{y' = f(t,y)} \Rightarrow$ 1st order ODE

nth order ODE

$$F(t, y, y', \ldots, y^{(n)}) = 0$$

$$\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ \vdots \\ y_n \end{pmatrix} = \begin{bmatrix} y \\ y' \\ \vdots \\ \vdots \\ y^{(n-1)} \end{bmatrix}$$

$$\Rightarrow \boxed{\vec{y}\,' = f(t, \vec{y})}$$

Existence - Peano

$f: R \times R \to R$ s.t it is continous in ~~$(-\delta, \delta)$~~

$(-\delta, \delta) \times (y_0 - \eta, y_0 + \eta)$ for some $\delta > 0$, $\eta > 0$

$\Rightarrow \exists\ \epsilon > 0$ and function $y \in C^1(-\epsilon, \epsilon)$ s.t

$y' = f(t, y)$; $y(0) = y_0$

Lipschitz continuity: $g: [a, b] \to R$

$L > 0$ $\quad |g(x) - g(y)| \leq L|x - y| \quad \forall x, y \in [a, b]$

Uniqueness - Cauchy-Lipschitz

$f(t, y) \to$ continous in both variables and Lipschitz continous in y variable.

$\exists\ \epsilon > 0$ and a unique function $y \in C^1(-\epsilon, \epsilon)$

s.t $y' = f(t, y)$; $y(0) = y_0$

Assume Lipshitz from now

$$y' = f(t, y);\ y(0) = y_0 \iff y(t) = y_0 + \int_0^t f(s, y(s))\,ds$$

Rather than solving integration at all $t \in [0,T]$ points we do at some points.

Mesh points

$N > 0$

$h = \frac{T}{N}$

$\boxed{t_n = nh}$ $n = 0, 1, \ldots N$

$y_n \approx y(t_n)$ $\cancel{\forall n}$ $h > 0$

$y_0 = y(t_0)$

$$y(t_{n+1}) = y(t_n) + \int_{t_n}^{t_{n+1}} f(s, y(s))\, ds$$

Eulers method

approximate $f(s, y(s))$ on $[t_n, t_{n+1}]$

$$f(s, y(s)) \approx f(t_n, y_n) \quad s \in [t_n, t_{n+1}]$$

Explicit

$$\boxed{y_{n+1} = y_n + h f(t_n, y_n)}$$

Trapezoidal method

Implicit

$$y_{n+1} = y_n + \frac{h}{2}\left(f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right)$$

$$y_{n+1} - \frac{h}{2} f(t_{n+1}, y_{n+1}) = y_n + \frac{h}{2} f(t_n, y_n)$$

Order

Let $y_{n+1} = \mathcal{F}(t, f, y_0, y_1, \ldots y_n, y_{n+1})$ be a reccurence relation, the order is p if

$$y(t_{n+1}) - \mathcal{F}(t, f, y(t_0), y(t_1) \ldots y(t_{n+1})) = O(h^{p+1})$$

Euler method

$$y(t_{n+1}) - y(t_n) - hf(t_n, y(t_n)) = y(t_n + h) - y(t_n) - hf(t_n, y(t_n))$$

$$= \cancel{y(t_n)} + \cancel{hy'(t_n)} + \frac{h^2}{2} y''(\xi) - \cancel{y(t_n)} - \cancel{hf(t_n, y(t_n))}$$

$$= \frac{h^2}{2} y''(\xi) = O(h^2)$$

$\boxed{\text{Order} = 1}$

Trapezoidal method

$$y(t_{n+1}) - y(t_n) - \frac{h}{2}(f(t_n, y(t_n)) + f(t_{n+1}, y(t_{n+1})))$$

$$= y(t_n + h) - y(t_n) - \frac{h}{2}(y'(t_n) + y'(t_n + h))$$

$$= y(t_n) + hy'(t_n) + \frac{h^2}{2} y''(t_n) + \frac{h^3}{6} y'''(\xi)$$
$$- y(t_n) - \frac{h}{2}\left(2y'(t_n) + hy''(t_n) + \frac{h^2}{2} y'''(\eta)\right)$$

$$= \frac{h^3}{6} y'''(\xi) - \frac{h^3}{4} y'''(\eta) = O(h^3)$$

$\boxed{\text{Order} = 2}$

Method of order p recoveres exactly every polynomial with degeree $\leq p$.

Global error: $\boxed{y_{n,h}} = y_n$ (New notation) $\qquad E_n = \max_{i=0,1,\dots,[\frac{T}{h}]} |e_{i,h}|$ (Global error)

$$e_{n,h} = y_{n,h} - y(t_n)$$

In $[0,T]$ there are $[\frac{T}{h}]+1$ ~~po~~ equally spaced mesh points

Convergence: A numerical method is convergent if

$$\lim_{h\to 0^+} \max_{n=0,1,\dots,[\frac{T}{h}]} |e_{n,h}| = 0 \quad \left(n=0,1,\dots,[\tfrac{T}{h}]\right)$$

Euler method

If $|e_{n,h}| \leq Ch^p$ then the convergence order is p. (also called order of Global error)

For Euler's method

$$e_{n+1,h} = y_{n+1,h} - y(t_{n+1})$$

$$= \cancel{y_{n,h} + hf(t_n, y_{n,h}) - y(t_n) - hf(t_n, y(t_n)) + o(h^2)}$$

$$= y_{n,h} + hf(t_n, y_{n,h}) - y(t_n) - hf(t_n, y(t_n)) + o(h^2)$$

$$= e_{n,h} + h\left(f(t_n, y_{n,h}) - f(t_n, y(t_n))\right) + Ch^2$$

$$\Rightarrow |e_{n+1,h}| \leq |e_{n,h}| + h|f(t_n, y_{n,h}) - f(t_n, y(t_n))| + Ch^2$$

$$\Rightarrow |e_{n+1,h}| \leq |e_{n,h}|(1+Lh) + Ch^2 \quad (L = \text{Lipshcitz constant})$$

$$\Rightarrow |e_{n,h}| \leq \frac{Ch}{L}\left((1+Lh)^n - 1\right) \quad (\text{By induction})$$

$$\Rightarrow |e_{n,h}| \le \frac{ch}{L}\left((1+Lh)^n - 1\right) \le \frac{ch}{L}\left(e^{LT}-1\right)$$

$\boxed{\text{order}=1}$

$$\Rightarrow \boxed{\lim_{h\to 0} |e_{n,h}| = 0} \quad \left((1+Lh)^n < e^{nLh} \le e^{[\frac{T}{h}]Lh} \le e^{LT}\right)$$

Trapezoid rule

$$e_{n+1,h} = e_{n,h} + \frac{h}{2}\left(f(t_n, y_{n,h}) - f(t_n, y(t_n))\right) + \frac{h}{2}\left(f(t_{n+1}, y_{n+1,h}) - f(t_{n+1}, y(t_{n+1}))\right) + O(h^3)$$

$$\Rightarrow |e_{n+1,h}| \le |e_{n,h}| + \frac{h}{2}L|e_{n,h}| + \frac{h}{2}L|e_{n+1,h}| + ch^3$$

$$\Rightarrow |e_{n+1,h}| \le \left(\frac{1+\frac{Lh}{2}}{1-\frac{Lh}{2}}\right)|e_{n,h}| + \left(\frac{1}{1-\frac{Lh}{2}}\right)ch^3$$

$$\Rightarrow |e_{n,h}| \le \frac{ch^2}{L}\left[\left(\frac{1+\frac{Lh}{2}}{1-\frac{Lh}{2}}\right)^n - 1\right] \quad n = 0, 1, \dots, \left[\frac{T}{h}\right]$$

$$\Rightarrow |e_{n,h}| \le \frac{ch^2}{L} e^{\frac{nLh}{1-\frac{Lh}{2}}} \le \frac{ch^2}{L} e^{\frac{TL}{1-\frac{L}{2}h}}$$

$$\Rightarrow \lim_{h\to 0} |e_{n,h}| = 0 \qquad \text{(convergence order=2)}$$

$\boxed{3C}$

One Step method: approximation at nth step depends on approx at (n-1)th step

General s-step method: $s \ge 1$

$$\sum_{k=0}^{s} \alpha_k y_{n+k} = h \sum_{k=0}^{s} \beta_k f_{n+k}$$

$$f_{n+k} = f(t_{n+k}, y_{n+k})$$

α_i, β_j are constants.

$$\boxed{\begin{array}{ll} \beta_s = 0 & \text{Explicit} \\ \beta_s \ne & \text{Implicit} \end{array}}$$

Need $y_0, \ldots, y_{s-1}$ to $\boxed{\text{start}}$

Sometimes these are generated using a 1-step method like Euler. $\boxed{\text{Order}=2}$

Adams' s-step method or (Adams - Bashforth) (Explicit) (Order S)

$t_n = nh$

Let $\Psi(t)$ be the polynomial interpolation with

$\Psi_i = f_i = f(t_i, y_i)$ $\quad i = n, n+1, \ldots, n+s-1$

$$\Psi(t) = \sum_{k=0}^{s-1} \Psi_k(t) f(t_{n+k}, y_{n+k}) \quad \text{(degree } s-1)$$

$$\Psi_k(t) = \prod_{l=0,\ l\neq k}^{s-1} \left(\frac{t - t_{n+l}}{t_{n+k} - t_{n+l}} \right)$$

Substitute $\Psi(t)$ for $f(t, y(t))$ in (t_{n+s-1}, t_{n+s})

$$y(t_{n+s}) - y(t_{n+s-1}) = \int_{t_{n+s-1}}^{t_{n+s}} f(\tau, y(\tau))\, d\tau$$

$$\Rightarrow \boxed{\begin{aligned} y_{n+s} - y_{n+s-1} &= h \sum_{k=0}^{s-1} \beta_k f(t_{n+k}, y_{n+k}) \\ \beta_k &= \frac{1}{h} \int_0^h \Psi_k(t_{n+s-1} + \tau)\, d\tau \end{aligned}}$$

(Independent of n)

$\boxed{\text{2 step}}$ $\quad y_{n+2} - y_{n+1} = \frac{h}{2}(3f_{n+1} - f_n)$

Adams Bashforth 3 step & 4 step

$\boxed{3}$ $\quad y_{n+3} = y_{n+2} + h\left(\frac{23}{12} f_{n+2} - \frac{16}{12} f_{n+1} + \frac{5}{12} f_n\right)$

$\boxed{4}$ $\quad y_{n+4} = y_{n+3} + h\left(\frac{55}{24} f_{n+3} - \frac{59}{24} f_{n+2} + \frac{37}{24} f_{n+1} - \frac{9}{24} f_n\right)$

$\boxed{1}$ $\quad y_{n+1} = y_n + h f_n$

(depends on s) $\beta_{s-j-1} = \cancel{\beta}_{s-j-1} = \dfrac{(-1)^j}{\lfloor j \lfloor s-j-1} \int_0^1 \prod_{\substack{i=0 \\ i \neq j}}^{s-1} (u+i)\, du$

$j = 0, \ldots s-1$

Adams-Moulton method (s step = s+1 order) (Implicit)

$$y_{n+s} = y_{n+s-1} + h \sum_{k=0}^{s} \beta_k \cancel{f_{n+s-k}} f_{n+k}$$

$$\beta_{s-j} = \cancel{\beta}_{s-j} = \frac{(-1)^j}{\lfloor j \lfloor s-j} \int_0^1 \prod_{\substack{i=0 \\ i \neq j}}^{s} (u+i-1)\, du \qquad j = 0, 1, \ldots s$$

$\Psi_i = f_i \quad i = n, n+1, \ldots, n+s$

s=0 $\quad y_n = y_{n-1} + hf(t_n, y_n)$

1 $\quad y_{n+1} = y_n + \frac{1}{2} h (f_{n+1} + f_n)$

2 $\quad y_{n+2} = y_{n+1} + h\left(\frac{5}{12} f_{n+2} + \frac{2}{3} f_{n+1} - \frac{1}{12} f_n\right)$

3 $\quad y_{n+3} = y_{n+2} + h\left(\frac{9}{24} f_{n+3} + \frac{19}{24} f_{n+2} - \frac{5}{24} f_{n+1} + \frac{1}{24} f_n\right)$

Order of multistep ~~Adams-Bashforth~~ method

$\sum_{k=0}^{s} \alpha_k y_{n+k} = h \sum_{k=0}^{s} \beta_k f_{n+k}$ is s-step and order p

$$\Rightarrow \quad \sum_{k=0}^{s} \alpha_k = 0$$

$$\sum_{k=0}^{s} k^m \alpha_k = m \sum_{k=0}^{s} k^{m-1} \beta_k \quad m = 1, 2, \ldots, p$$

$$\sum_{k=0}^{s} k^{p+1} \alpha_k \neq (p+1) \sum_{k=0}^{s} k^p \beta_k$$

Proof:

$$\sum_{k=0}^{s} \alpha_k y(t_{n+k}) - h \sum_{k=0}^{s} \beta_k f(t_{n+k}, y(t_{n+k}))$$

$$= \sum_{k=0}^{s} \alpha_k \left(y(t_n) + \sum_{m=1}^{\infty} \frac{k^m h^m}{m!} y^{(m)}(t_n) \right) - h \sum_{k=0}^{s} \beta_k y'(t_n + kh)$$

$$\left(y'(t_n + kh) = \sum_{m=0}^{\infty} (m+1) \frac{k^m h^m}{(m+1)!} y^{(m+1)}(t_n) \right.$$

$$= \sum_{m=1}^{\infty} m \frac{k^{m-1} h^{m-1}}{m!} y^{(m)}(t_n)$$

$$= \left(\sum_{k=0}^{s} \alpha_k \right) + \sum_{m=1}^{\infty} \left(\sum_{k=0}^{s} k^m \alpha_k - m \sum_{k=0}^{s} k^{m-1} \beta_k \right) \frac{h^m}{m!} y^{(m)}(t_n) =$$

$$= O(h^{p+1})$$

Convergence

1st characteristic polynomial: $\rho(z) = \sum_{k=0}^{s} \alpha_k z^k$

2nd characteristic polynomial: $\sigma(z) = \sum_{k=0}^{s} \beta_k z^k$

Dahlquist Equivalence: An s-step method is convergent iff
1) is of order $p \geq 1$
2) roots of $\rho(z)$ lie in the closed disk $|z| \leq 1$ with any which lie on the unit circle being $\boxed{\text{SIMPLE.}}$

Dahlquist's Barrier theorem: For s-step order cannot exceed
1) $s+1$ if s is ODD
2) $s+2$ if s is EVEN

Runge - Kutta Method

Explicit if A is strictly lower triangular

ν-stage

$$y_{n+1} = y_n + h \sum_{i=1}^{\nu} b_i f(t_n + c_i h, \xi_i)$$

$$\xi_i = y_n + h \sum_{j=1}^{i-1} a_{i,j} f(t_n + c_j h, \xi_j)$$

$A = (a_{i,j})$ is called RK Matrix

b_i = RK weights c_i = RK Nodes $\boxed{c_1 = 0}$.

RK Tableau

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array} = \begin{array}{c|ccccc} 0 & & & & & \\ c_2 & a_{21} & & & & \\ c_3 & a_{31} & a_{32} & & & \\ c_s & a_{s1} & a_{s2} & \cdots & a_{s,s-1} & \\ \hline & b_1 & b_2 & & b_{s-1} & b_s \end{array}$$

$\boxed{s = \nu}$

Consistent iff $\sum_{i=1}^{s} b_i = 1$

Popular: $\sum_{j=1}^{i-1} a_{ij} = c_i$

Standard RK or RK4

$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & 0 & 1 & \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array}$$

$y_{n+1} = y_n + \frac{1}{6} h (k_1 + 2k_2 + 2k_3 + k_4)$

$t_{n+1} = t_n + h$

$k_1 = f(t_n, y_n)$

$k_2 = f(t_n + \frac{h}{2}, y_n + h\frac{k_1}{2})$

$k_3 = f(t_n + \frac{h}{2}, y_n + h\frac{k_2}{2})$

$k_4 = f(t_n + h, y_n + hk_3)$

$\boxed{4A}$ ~~$M_{m,n}(\mathbb{R})$ = All m×n matrices.~~

$M_{m,n}(\mathbb{R})$ = All m×n matrices

$A = (A_{ij})_{1\le i\le m,\, 1\le j\le n} \in M_{m,n}(\mathbb{R})$

$A = A^i{}_j,\ X = X^j$

Objective

$AX = b$

$A \in M_{m,n}(\mathbb{R})\ X \in \mathbb{R}^n\ b \in \mathbb{R}^m$

$\boxed{A^i{}_j X^j = b^i}$

$\boxed{C^i{}_k = A^i{}_j B^j{}_k}$

Underdetermined	Overdetermined	
$\boxed{1}$ $m<n$	$\boxed{2}$ $m>n$	$\boxed{3}$ $m=n$
more unknows	more ~~soluti~~ equations	→ No solution
→ No solution	→ No solution	→ Single solution
→ Infinitely many	→ Single solution	→ Infinite
	→ Infinite	

$\boxed{\text{A is not invertible}} \Leftrightarrow \mathrm{Ker}(A) \neq \{0\} \Leftrightarrow \boxed{\text{A is singular}}$

$\mathrm{Ker}(A) = \{v \in V \mid Av = 0\}$

No solution or ∞

If $Ay = b$ then $Ax = b$

$x = y + c\eta \quad c \in \mathbb{R}\ \eta \in \mathrm{Ker}(A) - \{0\}$

For square matriced

$\underbrace{\mathrm{Rank}(A)}_{\text{dimension of rowspace}} + \underbrace{\mathrm{nullity}(A)}_{\text{dimension of kernel or null space}} = n$

$\boxed{\text{A is invertible}}$

$x = A^{-1} b$ (Hard)

Cramer's rule

$x_i = \dfrac{\det(A_i)}{\det(A)}$ A_i is obtained by replacing ith column of A by b.

Diagonal matrices $\boxed{x_i = \dfrac{b_i}{A_{ii}}}$

Unitary matrix $(A^{-1} = A^T)$ (orthogonal $\Rightarrow A^T = A^{-1}$) (Unitary $\Rightarrow A^* = A^{-1}$)

$\boxed{|\lambda| = 1}$

$$x_i = \sum_{j=1}^{n} A_{ji} b_j$$

Upper triangular Matrix $(A^i_j = 0 \ \forall\ i > j)$

$$x_i = \frac{b_i - \sum_{j=i+1}^{N} a_{ij} x_j}{a_{ii}}$$

(Back substitution Algorithm)

$$x_n = \frac{b_n}{A_{nn}}$$

(similarly for lower)

Gauss method (Only for invertible matrices)

$\{A^{(1)}, \dots, A^{(n)}\}$ $\{b^{(1)}, \dots, b^{(n)}\}$

$A^{(1)} = A$ $A^{(n)} =$ Upper triangular

$$A^{(k+1)}_{ij} = A^{(k)}_{ij} - m_{ik} A^{(k)}_{kj} \quad \boxed{\text{for } i = k+1, \dots, n; \ j = 1, \dots, n}$$

$$b^{(k+1)}_i = b^{(k)}_i - m_{ik} b^{(k)}_k \quad \boxed{\text{for } i = k+1, \dots, n}$$

$$\boxed{m_{ik} = \frac{A^{(k)}_{ik}}{A^{(k)}_{kk}}}$$

Theorem:

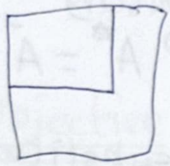
Let A be a $n \times n$ matrix. There exists at least one nonsingular matrix B, such that the product BA is Upper-Triangular.

$$BAx = Bb$$

For non-Invertible matrices the dignol contains atleast 1 zero element.

Pivoting: If matrix is invertible but $A_{ii} = 0$ for some i, we need to do pivoting.

Diagnol submatrices



$$\Delta^k = \begin{pmatrix} A_{11} & \cdots & A_{1k} \\ \vdots & \ddots & \vdots \\ A_{k1} & \cdots & A_{kk} \end{pmatrix}$$

Theorem: Let all diagonal submatrices of order $k = 1, 2, \ldots n$ $\boxed{\text{are invertible}}$ for A. Then Gaussian elemination doesn't need any pivoting strategy.

$\boxed{4B}$

Factorization $A = BC$

$$B(Cx) = b$$

$$B\,y = b \quad A\varepsilon \quad y = Cx$$

$\boxed{L^i_i = 1} \Rightarrow$ Unique

LU factorization $(A = LU)$ It should be

LU factorization $\Leftrightarrow$ Gaussian elimination without pivoting.

$$U = A^{(n)}$$

$$L_{ij} = \begin{cases} 1 & i = j \\ m_{ij} & i > j \\ 0 & i < j \end{cases}$$

$$U_{1j} = A_{1j}$$

$$U_{ij} = A_{ij} - \sum_{k=1}^{j-1} L_{ik} U_{kj}$$

$\boxed{\begin{array}{l} i = 1, 2, \ldots n \\ j = i, \ldots, n \end{array}}$

$$L_{j+1,j} = \frac{A_{j+1,j} - \sum_{k=1}^{j-1} L_{j+1,k} U_{kj}}{U_{jj}}$$

$$L_{ij} = \frac{A_{ij} - \sum_{k=1}^{j-1} L_{ik} U_{kj}}{U_{jj}} \quad i = 2, \dots, n \quad j = 1, \dots, i-1$$

$$\boxed{A_{ij} = \sum_{k=1}^{\min(i,j)} L_{ik} U_{kj}}$$

Cholesky Factorisation $(A = BB^T)$ or LL^T B = lower triangular Unique

$\boxed{\langle Ay, y \rangle = A_{ij} y^i y^j}$ Only for Hermitian and positive definite matrices.

Hermitian → symmetric

Positive definite $\Leftrightarrow$ all $\lambda_i > 0$

Positive semidefinite $\Leftrightarrow$ $\lambda_i \geq 0$

$\forall y \neq 0$
$\langle Ay, y \rangle > 0$
$\langle Ay, y \rangle \geq 0$

$$B_{jj} = \sqrt{A_{jj} - \sum_{k=1}^{j-1} B_{j,k} B^*_{j,k}}$$

$$B_{ij} = \frac{1}{B_{jj}} \left(A_{ij} - \sum_{k=1}^{j-1} B_{i,k} B^*_{j,k} \right) \text{ for } i > j$$

For strictly positive semidefinite $B_{jj} = 0$ for some j and the method fails.

For indefinite matrix $\Rightarrow B^2_{jj} < 0$ for some j

Test for +Ve definiteness: Fails if $B_{jj} = 0$ or $B^2_{jj} < 0$

QR Factorization $(A = QR)$ or QU (for any ~~method~~ matrix)

Q = Orthogonal $(Q^T = Q^{-1})$

R = Upper triangular

Theorem: If $\det(A) \neq 0 \Rightarrow A = QR$ such that $\det(R) \neq 0$.

If R is assumed to have +ve diagonal entries then (Q, R) is unique.

Algorithm Let $A_1, A_2 \ldots A_n$ be columns of A.

q_i is a cloumn

$$q_i = \frac{A_i - \sum_{k=1}^{i-1} \langle q_k, A_i \rangle q_k}{\left\| A_i - \sum_{k=1}^{i-1} \langle q_k, A_i \rangle q_k \right\|}$$

$$\boxed{A_i = \sum_{k=1}^{n} R_{ki} q_k}$$

$$R_{ki} = \begin{cases} \langle q_k, A_i \rangle & \text{if } 1 \leq k \leq i-1 \\ \left\| A_i - \sum_{k=1}^{i-1} \langle q_k, A_i \rangle q_k \right\| & \text{if } k = i \\ 0 & \text{if } k > i \end{cases}$$

or

$$R_{ki} = \begin{cases} \langle q_k, A_i \rangle & \text{if } 1 \leq k \leq i \\ 0 & \text{if } k > i \end{cases}$$

4C Norm on $\mathbb{R}^n$

1) $||x|| > 0$ $\forall x \neq 0$ 2) $||\alpha x|| = |\alpha| \, ||x||$ 3) $||x+y|| \leq ||x|| + ||y||$

l^p-norm for $p \geq 1$

$$||x||_{l^p} = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$$

l^∞ norm

$$||x||_{l^\infty} = \max_{1 \leq i \leq n} |x_i|$$

$$\boxed{||x||_{l^2} = ||x||_2 = \sqrt{\langle x, x \rangle}}$$

Equivalence

$$m||x|| \leq ||x||' \leq M||x|| \quad \forall x \in \mathbb{R}^n$$

Independent of x

$$\frac{1}{M}||x||' \leq ||x|| \leq \frac{1}{m}||x||'$$

→ All norms are equivalent of $\mathbb{R}^n$

$$||x||_{l^\infty} \leq ||x||_{l^p} \leq \sqrt[p]{n} \, ||x||_{l^\infty}$$

$$||x||_{l^2} \leq ||x||_{l^1} \leq \sqrt{n} \, ||x||_{l^2}$$

Matrices

$$||A||_{l^p} = \left(\sum_{j=1}^{n} \sum_{m=1}^{m} |A_{ij}|^p \right)^{\frac{1}{p}}$$

$\boxed{p = 2 = \text{Forbenius norm}}$

$$||A||_{l^\infty} = \max_{1 \leq i \leq m, \, i \leq j \leq n} |A_{ij}|$$

$$||A||_{L_{p,q}} = \left(\sum_{j=1}^{n} \left(\sum_{i=1}^{m} |A_{ij}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}}$$

$$||AB||_{l^2} \leq ||A||_{l^2} ||B||_{l^2}$$

Matrix Norm or **Submultiplicative norm**

③ $+ \; ||AB|| \leq ||A|| \, ||B|| \quad \forall A, B \in M_n(\mathbb{R})$

Ex: $\boxed{||AB||_{l^2} \text{ or } ||\cdot||_{l^2}}$ $\boxed{1 \leq ||I_n||}$

$||\cdot||_{l^\infty}$ is not a matrix norm.

Subordinate matrix norm: $\|\cdot\|$ be a vector norm.

$$\|A\| = \sup_{x\in\mathbb{R}^n\setminus\{0\}} \frac{\|Ax\|}{\|x\|} = \sup_{\forall \|y\|=1} \|Ay\|$$

$\Rightarrow \quad \|Ax\| \le \|A\|\,\|x\|$ & $\|AB\| \le \|A\|\,\|B\|$

Proof: $\|AB\| \le \|A\| \sup_{\|x\|=1} \frac{\|Bx\|}{\|x\|} \le \|A\|\,\|B\|$

Subordinate matrix norm $\Rightarrow \|I_n\| = 1$

Forbenius norm is not subordinate. $\boxed{\|I_n\|_{\ell^2} = \sqrt{n}}$ $\longrightarrow$ But vector norm.

$$\|A\|_p = \sup_{\|x\|=1} \frac{\|Ax\|_p}{\|x\|_p}$$

$\|\cdot\|_2$ forbenius subordinate

1) A is unitary $(A^{-1} = A^T) \Rightarrow \|A\|_2 = 1$

2) $A =$ Unitary $\Rightarrow \|AB\|_2 = \|BA\|_2 = \|B\|_2$

due to $\|x\|_2 = \sqrt{\langle x, x\rangle}$

Diagonal matrix $\Rightarrow \|A\|_2 = \max_{1\le i\le n} |a_i|$

$\boxed{\text{4D}}$

Normal Matrix : $AA^* = A^*A$

$\Rightarrow \quad \exists\, U$ s.t U is unitary and

$$\boxed{A = U \operatorname{diag}(\lambda_1, \dots \lambda_n)\, U^*}$$

If $A =$ normal $\Rightarrow \|A\|_2 = \|\operatorname{diag}(\lambda_1 \dots \lambda_n)\|_2$

$$= \max_{1\le i\le n} |\lambda_i|$$

$$= \rho(A)$$

Spectral radius : For any matrix A

$$\varrho(A) = \max_{1 \le i \le n} |\lambda_i|$$

$\downarrow$ not a matrix norm.

$\rightarrow$ for any matrix norm $\quad \varrho(A) \le \|A\|$

Converse : for any A $\exists$ a subordinate matrix

such that $\|A\| \le \varrho(A) + \epsilon$

$\|\cdot\|$ depends on A & ϵ.

$$\|A\|_1 = \max_{1 \le j \le n} \left(\sum_{i=1}^{n} |A_{ij}| \right) \quad \text{(column sum)}$$

$$\|A\|_\infty = \max_{1 \le i \le n} \left(\sum_{j=1}^{n} |A_{ij}| \right) \quad \text{(Row sum)}$$

$$\Rightarrow \quad \varrho(A) \le \min \left(\|A\|_1 , \|A\|_\infty \right)$$

Hermitian $(A = A^*)$: iff $A = U \operatorname{diag}(\lambda_1, \dots \lambda_n) U^*$ & all λ's are real.

$\rightarrow$ For any A, AA^* is Hermitian.

Singular values : of A are the nonnegative square roots of AA^* eigen values.

$\rightarrow$ remove zeroes.

$\rightarrow$ For A = normal $\Rightarrow A = UDU^* \Rightarrow A^*A = UD^*DU^*$

So singular values are $\{|\lambda_1|, |\lambda_2| \dots |\lambda_n|\}$

$\boxed{\text{Subordinate } \|A\|_2}$ For any matrix A

λ_i are eigenvalues of D.

$$\|A\|_2 = \max_{1 \le i \le n} \sqrt{\lambda_i} = \text{largest singular value of } A.$$

$A^* A = U D U^*$ / Both have $\lambda_1, \lambda_2, \ldots, \lambda_n$ eigen values. 22

$\boxed{4E}$

Convergence for any Norm (since all are equivalent)

$$\lim_{k\to\infty} A^{(k)} \to A \iff \lim_{k\to\infty} \|A^{(k)} - A\| = 0$$

Theorem i), ii) iii) & iv) are equivalent.

i) $\lim_{k\to\infty} A^k = 0$ ii) $\lim_{k\to\infty} A^k x = 0 \ \forall x \in \mathbb{C}^n$

iii) $\varrho(A) < 1$ iv) $\exists$ a subordinate matrix norm suct that $\|A\| < 1$

Geometric series: $\sum_{k=0}^{\infty} A^k$ converges iff $\varrho(A) < 1$.

$\triangleright$ Any invertible matrix has a neighbourhood in which there are invertible matrices.

$$\|A - B\| < \frac{1}{\|A^{-1}B\|}$$

$$\Rightarrow \varrho(A^{-1}(A-B)) < 1$$

$$\Rightarrow B = A(I - (A^{-1}(A-B))) \text{ is invertible.}$$

Relative error $\frac{\|\tilde{y} - y\|}{\|y\|}$ (depends on $\|\cdot\|$)

Condition Number $\operatorname{cond}(A) := \|A\| \|A^{-1}\|$

relative to subordinate matrix norm $\|\cdot\|$

$$\operatorname{cond}(A) \geq 1$$

$$(A+B)^{-1} = (I + A^{-1}B)A^{-1}$$

Error

$$x = A^{-1} b \qquad A_\epsilon = A + \epsilon B$$

$$x_\epsilon = A_\epsilon^{-1} b_\epsilon \qquad b_\epsilon = b + \epsilon C$$

$$\frac{\|x_\epsilon - x\|}{\|x\|} \leq \text{cond}(A) \left\{ \frac{\|b_\epsilon - b\|}{\|b\|} + \frac{\|A_\epsilon - A\|}{\|A\|} \right\} \rho$$

Well conditioned $\Rightarrow$ cond $(A) \simeq 1$ ILL conditioned $\Rightarrow$ cond$(A) >> 1$

$\text{Cond}_2(A) = 1$ for any Unitary A

$$\text{Cond}(A) = \|A\| \, \|A^{-1}\| \geq \rho(A)\, \rho(A^{-1})$$

If A = normal

$$\Rightarrow \text{Cond}_2(A) = \frac{|\lambda_{max}|}{|\lambda_{min}|}$$

If A = any invertible

$$\Rightarrow \text{Cond}_2(A) = \frac{\mu_{max}}{\mu_{min}}$$

$\boxed{4F}$ Iterative method converges if for any choice of $x^{(0)} \in \mathbb{R}^n$, $x^{(k)}$ converges to x.

$$r^{(k)} = Ax^{(k)} - b = Ae^{(k)}$$

$$e^{(k)} = x^{(k)} - x$$

$$\lim_{k \to \infty} x^{(k)} - x = \lim_{k \to \infty} r^{(k)} = 0$$

Splitting: $M =$ easily Invertible & $A = M - N$

$$x^{(k+1)} = M^{-1} N x^{(k)} + M^{-1} b$$

Conveges

$$x^{(k+1)} = M^{-1} N x^{(k)} + M^{-1} b$$

converges iff $\varrho(M^{-1}N) < 1$

$$e^{(k)} = M^{-1} N e^{(k-1)}$$

Estimate

$$\| e^{(k)} \| \sim [\varrho(M^{-1}N)]^k \| e^{(0)} \| \text{ for } k \gg 1$$

Richard's iterative method (steepest descent/gradrent)

$$M = \frac{1}{\alpha} I_n \quad N = \frac{1}{\alpha} I_n - A$$

$$M^{-1} N = I_n - \alpha A$$

$$\boxed{|1 - \alpha \lambda_i| < 1 \ \forall i}$$

If $A =$ real & symmetric & $\lambda_1 \le \lambda_2 \quad \le \lambda_n$

→ $\lambda_i \lambda_j < 0$ for some i, j ⇒ Richardson's method doesn't converge.

→ $\lambda_i > 0 \ \forall i$ then it converges iff $0 < \alpha < \frac{2}{\lambda_n}$

→ $\lambda_i < 0 \ \forall i$ then 〃 iff $\frac{2}{\lambda_1} < \alpha < 0$

Optimal α: Let A be a matrix with +ve eigenvalues

$$\Rightarrow \varrho(I - \alpha A)_{\min} \Rightarrow \tilde{\alpha} = \frac{2}{\lambda_1 + \lambda_n}$$

$$\varrho(I - \tilde{\alpha} A) = \frac{\lambda_n - \lambda_1}{\lambda_n + \lambda_1}$$

$$e(I_n - \tilde{\alpha} A) = \frac{\text{Cond}_2(A) - 1}{\text{Cond}_2(A) + 1}$$

well conditioned $\Rightarrow$ fast Richardson convergence.

$\boxed{4G}$

Jacobi method: $M = D = \text{diag}(A_{11}, \dots A_{nn})$

$N = D - A \quad M^{-1}N = I_n - D^{-1}A$

$|A_{ii}| > \sum_{k=1, k\neq i}^{n} |A_{ik}| \Leftrightarrow$ strictly row-diagnally dominant

$\Leftrightarrow$ strictly column-diagnally dominant

$|A_{ii}| > \sum_{k=1, k\neq i}^{n} |A_{ki}|$

If A is either strictly row or column diagnol dominant then Jacobi converges.

Gauss-Seidel method

$D = \text{diag}(A_{11}, A_{22}, \dots A_{nn})$

$F_{ij} = \begin{cases} -A_{ij} & i<j \\ 0 & i\geq j \end{cases} \Bigg| \quad E_{ij} = \begin{cases} -A_{ij} & \text{for } i>j \\ 0 & \text{otherwise} \end{cases}$

$A = D - E - F$

$M = D - E \quad N = F$

$M^{-1}N = (D(I_n - D^{-1}E))^{-1} F = (I_n - D^{-1}E)^{-1} D^{-1} F$

$L = D^{-1}E \quad U = D^{-1}F$

$$\boxed{G = M^{-1}N = (I_n - L)^{-1} U}$$

For Jacobi method $J = I_n - D^{-1}A = I_n - D^{-1}(D-E-F)$

$$\boxed{J = L + U}$$

If A is strictly row-diagonally dominant matrix then Gauss-Seidel method converges.

$$\|G\|_\infty \leq \|J\|_\infty < 1$$

Theorem: If $J_{ij} \geq 0$, then only one of the following is true

1) $e(G) = e(J) = 0$
2) $0 < e(G) < e(J) < 1$
3) $e(G) = e(J) = 1$
4) $e(G) > e(J) > 1$

i.e if $J_{ij} \geq 0$ either both converge or diverge. When both converge Gauss-Seidel is better.

$\boxed{4H}$

Relaxed Gauss-Seidel

$$A = \frac{1}{\alpha} D - E + \left(1 - \frac{1}{\alpha}\right) D - F$$

$$M = \frac{1}{\alpha} D - E \quad , \quad N = \left(\frac{1}{\alpha} - 1\right) D + F$$

$$G_\alpha = \left(\frac{1}{\alpha} I - L\right)^{-1} \left(\left(\frac{1-\alpha}{\alpha}\right) I + U\right)$$

$$\boxed{e(G_\alpha) \geq |\alpha - 1|}$$

$$\Downarrow$$

convergence $\longrightarrow$ $\alpha \in (0,2)$

Let A be Hermitian $(A = A^*)$ & positive definite matrix. Also assume $M + M^* - A$ is positive definite.

λ is eigenvalue of $\boxed{H = A^{-1}(2M - A)}$

$$Re(\lambda) > 0$$

Also

$$M^{-1}N = (H-I)(H+I)^{-1}$$

and $e(M^{-1}N) < 1$

Proof

Let $M^{-1}NZ = \mu Z$

Let $y = (H+I)^{-1}Z$

$$\Rightarrow Hy = \left(\frac{1+\mu}{1-\mu}\right)y$$

$$\Rightarrow \mu = \frac{\lambda-1}{\lambda+1} \Rightarrow |\mu|^2 = \frac{|\lambda|^2+1-2Re(\lambda)}{|\lambda|^2+1+2Re(\lambda)} < 1$$

→ For symmetric positive definite matrices

$A \in M_n(\mathbb{R})$ $\quad e(G_\alpha) < 1 \quad \forall \alpha \in (0,2)$

Proof:

$$M + M^T - A = \left(\frac{2}{\alpha}-1\right)D$$

$$\text{Trace}(A) = \sum_{i=1}^{n} \lambda_i$$

$$\det(A) = \prod_{i=1}^{n} \lambda_i$$

Similar matrices $B = P^{-1}AP$ (A, B same eigen values)

For orthogonal $|\lambda| = 1$

Let $Ax = \lambda x$ and $|x_i| \geq |x_j| \ \forall j$

$$\Rightarrow \sum_{j \neq i} A_{ij} x_j = (\lambda - A_{ii}) x_i$$

$$\Rightarrow \boxed{|\lambda - A_{ii}| \leq \sum_{j\neq i} |A_{ij}|}$$

$D_i = \{z : |z - A_{ii}| \le R_i\}$ $\boxed{R_i = \sum_{j \ne i} |A_{ij}|}$

Any $\boxed{\lambda \in \bigcup_{i=1}^{n} D_i}$

$\boxed{5A}$

No formular for polynomial with degree ≥ 5

Fixed point if $\varphi(y) = y$

Write $f(x) = 0$ as $x - g(x) = 0$

<u>Fixed point theorem:</u> continous $g : [a, b] \to [a, b]$

$\exists\ \xi \subset [a, b]$ s.t $\xi = g(\xi)$

<u>Proof:</u> $g(a) - a \ge 0$ & $g(b) - b \le 0$

<u>Iteration</u> $x^{(k+1)} = g(x^{(k)})$

<u>Contraction</u> continous $g : [a,b] \to \mathbb{R}$ is a contraction

on $[a,b]$ if $L \in (0,1)$

$|g(x) - g(y)| \le L\,|x - y|$

$\to$ If g is a contraction on $[a,b]$ it has a unique fixed point on $[a,b]$ & any $x^{(0)} \in [a,b]$ will <u>converge</u>. (order=1 $c<1$)

$$\boxed{|x^{(k)} - \xi| \le L^k\,|x^{(0)} - \xi|}$$

$\Rightarrow$ If g' is continous on $[\xi-h, \xi+h]$

and $|g'(\xi)| < 1$ then converges for

$x^{(0)} \in (\xi - \delta, \xi+\delta)$ s.t $|g'(x)| \leq L < 1 \ \forall x \in [\xi-\delta, \xi+\delta]$

Relaxation ~~zero~~ iteration

$$x^{(k+1)} = x^{(k)} - \lambda f(x^{(k)})$$

$$g(x) = x - \lambda f(x)$$

$$f(x) = 0 \Leftrightarrow g(x) = x$$

$\exists$ $\lambda > 0$ & $\delta > 0$ s.t it converges if $f'(\xi) \neq 0$

$x^{(0)} \in [\xi - \delta, \xi + \delta]$ where $|f'(x)| < 1$

$\forall x \in (\xi - \delta, \xi + \delta)$

$\boxed{\text{SB}}$ Error : $e^{(k)} = x^{(k)} - \xi$

Order of convergence If $\exists$ $C > 0$ & $p \geq 1$

(with $C < 1$ if $p = 1$) and ~~and~~ an iteger

N s.t for all $k \geq N$

$$|e^{(k+1)}| \leq C |e^{(k)}|^p \qquad (C < 1 \text{ if } p = 1)$$

Generalised relaxation iteration

$$x^{(k+1)} = x^{(k)} - \lambda(x^{(k)}) f(x^{(k)})$$

$$g(x) = x - \lambda(x) f(x)$$

convergence if $|1 - \lambda(\xi) f'(\xi)| < 1$

Newton's method: $$x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})}$$

$$g(x) = x - \frac{f(x)}{f'(x)} \qquad f'(x^{(k)}) \neq 0 \ \forall k \geq$$

Convergence If $f : [a,b] \to \mathbb{R}$ is C^2 on $I_\delta = [\xi-\delta, \xi+\delta]$ for $\delta > 0$. Let $f(\xi) = 0$, $f'(\xi) \neq 0$ & $f''(\xi) \neq 0$.

If $\frac{|f''(x)|}{|f'(y)|} \leq M \ \forall x, y \in I_\delta$

and if $|x^{(0)} - \xi| \leq h$ with $h = \min\{\delta, \frac{1}{M}\}$

then it converges with order 2.

$$\lim_{k \to \infty} \left| \frac{\xi - x^{(k+1)}}{(\xi - x^{(k)})^2} \right| \leq \frac{M}{2} \qquad (p=2)$$

Zeroes with multiplicies

If $m > 1$ is the multiplicity.

$$x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})}$$ still converges but only $\boxed{\text{linearly}}$ $(p=1)$

$$x^{(k+1)} = x^{(k)} - m\frac{f(x^{(k)})}{f'(x^{(k)})}$$

ξ

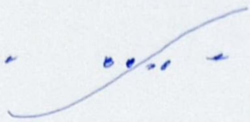
$$x^{(k+1)} = x^{(k)} - \frac{k(x^{(k)})}{k'(x^{(k)})}. \qquad \boxed{k(x) = \frac{f(x)}{f'(x)}}$$

converge with $\boxed{p=2}$

$$\boxed{g(x) = x - \frac{f(x)}{f'(x)}} \quad \boxed{g'(\xi) = 1 - \frac{1}{m} < 1}$$ linear convergence

$$f(x) = (x-\xi)^m \psi(x)$$

Bisection method



$\boxed{\text{slow}}$

Secant method

$$x_{k+1} = x_k - f(x_k)\left(\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}\right)$$

$$f'(\xi) \neq 0 \Rightarrow \frac{|\xi - x_{k+1}|}{|\xi - x_k|} \le \frac{2}{3}$$

$\boxed{p = 1 \text{ for convergence}}$

End